Enlarged terminal sets guaranteeing stability of receding horizon control

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Abstract

It is known that stability of a model predictive control system is ensured if the terminal conditions of the optimal control problem solved online satisfy certain criteria. The usual requirement is that the terminal cost function is a control Lyapunov function defined on the terminal constraint set. Conventionally the terminal cost function is chosen, when the system being controlled is linear, to be the value function for the infinite-horizon *unconstrained* optimal control problem and the terminal constraint set is chosen to be the output admissible set for the closed-loop system using the optimal unconstrained controller $u = -Kx$. The purpose of this paper is to relax these terminal conditions thereby facilitating online solution of the optimal control problem. Using some recent results, we present alternative conditions that employ, as the terminal cost, the finite-horizon cost resulting from a *nonlinear* controller $u = -\text{sat}(Kx)$ and, as the terminal constraint set, the set in which this controller is optimal for the finite-horizon *constrained* optimal control problem. It is shown that this solution provides a considerably larger terminal constraint set.

1 Introduction

This paper is concerned with closed-loop stability of constrained linear systems when model predictive control is employed. Model predictive control is a form of control in which the current control is obtained by solving, at each sampling instant, a finite-horizon open-loop optimal control problem and applying the first element of the optimal control sequence so obtained. Obviously, model predictive control of constrained systems is nonlinear so that stability is, in general, a nontrivial issue. After the pioneering work of Chen and Shaw [1], and of Keerthi and Gilbert [5], the value function of the finite-horizon optimal control problem has been used, almost universally, as a Lyapunov function for analyzing closed-loop stability [7].

Several 'ingredients' of the online optimal control problem directly affect closed-loop stability; these are: the terminal cost $F(\cdot)$, the terminal constraint set $\mathcal{X}_f$ (both of which are employed in the optimal control problem solved online), and the local controller $\kappa_f(\cdot)$ that ensures existence of feasible solutions to the optimal control problem (see, e.g., [1, 5, 7]). Ideally, the terminal cost $F(\cdot)$ is the infinite-horizon value function $V^0_\infty(\cdot)$ (for the constrained optimal control problem), in which case, the finite-horizon value

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function is $V_N^0(\cdot) = V_\infty^0(\cdot)$. With this choice, online optimisation is unnecessary and the advantages of an infinite-horizon problem automatically accrue. However, constraints generally render this approach impossible. Usually, then, $\mathcal{X}_f$ is chosen to be an appropriate neighbourhood of the origin in which $V_\infty^0(\cdot)$ is exactly (or approximately) known, and $F(\cdot)$ is set equal to $V_\infty^0(\cdot)$ or its approximation. When the system being controlled is linear, $F(\cdot)$ is often chosen (see [9, 10]) to be the value function of the infinite-horizon, *unconstrained* optimal control problem, $\kappa_f(\cdot)$ is chosen to be the optimal controller ($\kappa_f(x) = -Kx$) for this problem, and $\mathcal{X}_f$ the *maximal output admissible set* $\mathcal{O}_\infty$ (defined by (4.5) below) for the closed-loop system using the local controller $\kappa_f(\cdot)$. In this case, $V_\infty^0(x) = F(x) = x^T P x$ for all $x \in \mathcal{X}_f$. (P and K are given by the solution of an algebraic Riccati equation, see (3.13), (3.14) below.)

The purpose of this paper is to provide *new* terminal ingredients for model predictive control of *input constrained linear systems*. The ingredients are an improvement over those previously used in that the terminal constraint set $\mathcal{X}_f$ is strictly larger than $\mathcal{O}_\infty$, thus facilitating the solution of the optimal control problem. To obtain the improved terminal conditions we employ recent results [2] that show that the nonlinear controller $\kappa_{n\ell}(x) = -\text{sat}(Kx)$ is optimal in a region which includes the maximal output admissible set $\mathcal{O}_\infty$. The proposed terminal cost function $F(\cdot)$ is the finite-horizon value function $V_N^0(\cdot)$. The proposed terminal cost function $F(\cdot)$, while still convex (thus ensuring solvability of the optimal control problem) is no longer quadratic (which implies that the problem is not a quadratic program, and needs to be solved using convex programming or conventional nonlinear programming). A previous result [6] dealt with the case of *open-loop stable* linear plants. The present paper extends the previous result and is valid for arbitrary linear plants with constrained single input.

The structure of the paper is as follows. Some preliminary definitions and notation are introduced in §2. Model predictive control is briefly described in §3 where properties of the terminal conditions that ensure closed-loop stability are specified. In that section we also quote some recent results [2] that give a regional characterisation of the value function and optimal controller for a finite-horizon, constrained, optimal control problem; and the region in which this characterisation is valid. These results are used in §4 to provide new, improved terminal conditions for the model predictive controller.

2 Definitions and notation

The system considered is

$$x(k+1) = Ax(k) + Bu(k) \tag{2.1}$$

or, more concisely,

$$x^+ = f(x, u) := Ax + Bu$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are, respectively, the current state and control and x^+ is the successor state. The pair (A, B) is assumed controllable. The control is required to satisfy the constraint

$$u(k) \in \Omega \tag{2.2}$$

for all k, where $\Omega := [-1, 1]$.

The following notation will be employed. The solution of (2.1) at time k, when the initial state is x at time i and the control sequence is $\mathbf{u}$, is $x^{\mathbf{u}}(k; x, i)$; to simplify notation, $x^{\mathbf{u}}(k; x) := x^{\mathbf{u}}(k; x, 0)$, i.e. the initial time is dropped when it is zero. For all $\epsilon > 0$, $B_\epsilon := \{x \mid |x| \leq \epsilon\}$. For any set X in, say, $\mathbb{R}^n$, X^c denotes the complement of X (in $\mathbb{R}^n$). $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ and for each $\Delta \in \mathbb{R}^+$, the function

$\mathrm{sat}_\Delta(\cdot)$ is defined by

$$\mathrm{sat}_\Delta(u) := \begin{cases} u & \text{if } |u| \leq \Delta \\ \Delta & \text{if } u > \Delta \\ -\Delta & \text{if } u < -\Delta \end{cases} \tag{2.3}$$

The function $\mathrm{sat}(\cdot)$ is defined to be $\mathrm{sat}_1(\cdot)$. In the sequel $\circ$ denotes concatenation, i.e. $(a \circ b)(x) := a(b(x))$, $a^0(x) := x$ and, for all $i = 1, 2, \ldots$, $a^i(x) := (a^{i-1} \circ a)(x) = (a \circ a^{i-1})(x)$.

3 Model predictive control

In model predictive control, a finite-horizon optimal control problem $\mathcal{P}_N(x)$ defined below is repeatedly solved. Because of time invariance, the initial time in the optimal control problem may be taken to be zero. Thus $\mathcal{P}_N(x)$ is defined by

$$\mathcal{P}_N(x): \quad V_N^0(x) = \min_{\mathbf{u}} V_N(x, \mathbf{u}) \tag{3.1}$$

subject to the control constraint

$$\mathbf{u} \in \Omega^N \tag{3.2}$$

and the terminal constraint

$$x(N) \in \mathcal{X}_f \tag{3.3}$$

where

$$\mathbf{u} := \{u(0), u(1), \ldots, u(N-1)\} \tag{3.4}$$

is a sequence of N control actions,

$$V_N(x, \mathbf{u}) := \sum_{k=0}^{N-1} \ell(x(k), u(k)) + F(x(N)) \tag{3.5}$$

$$\ell(x, u) := |x|_Q^2 + |u|_R^2 \tag{3.6}$$

and $x(k) := x^{\mathbf{u}}(k; x)$, $k = 0, 1, \ldots, N$. We assume that Q and R are positive definite, and denote $|x|_Q^2 := x^T Q x$ and similarly for $|u|_R^2$. At event (x, k) (at state x, time k), problem $\mathcal{P}_N(x)$ is solved yielding the optimal control sequence

$$\mathbf{u}^0(x) = \{u^0(0; x), u^0(1; x), \ldots, u^0(N-1; x)\} \tag{3.7}$$

the optimal state sequence

$$\mathbf{x}^0(x) = \{x^0(0; x), x^0(1; x), \ldots, x^0(N; x)\} \tag{3.8}$$

(where $x^0(0; x) = x$, the initial state) and the value function

$$V_N^0(x) = V_N(x, \mathbf{u}^0(x)) \tag{3.9}$$

The first control $u^0(0; x)$ is applied to the plant so that the (implicit) model predictive control law is

$$u = \kappa_N(x) := u^0(0; x), \tag{3.10}$$

and the procedure is repeated as a new state becomes available.

3.1 Closed loop stability

If $F(\cdot)$ and $\mathcal{X}_f$ are chosen appropriately (see, for example, [7]), the receding horizon control law (3.10) can be shown to be stabilising. For any function $\theta : I\!R^n \to I\!R^n$, let $\overset{*}{\theta}(\cdot)$ be defined by

$$\overset{*}{\theta}(x,u) := \theta(f(x,u)) - \theta(x) \tag{3.11}$$

where $f(x,u) = Ax + Bu$. We have [7]:

Theorem 1 *Suppose the terminal cost function $F : \mathcal{X}_f \to I\!R$, the terminal constraint set $\mathcal{X}_f$ and the local control law $\kappa_f : \mathcal{X}_f \to I\!R$ satisfy:*

A1: *$\mathcal{X}_f$ is closed and $0 \in \mathcal{X}_f$,*

A2: *$\kappa_f(x) \in \Omega$, $\forall x \in \mathcal{X}_f$ (control constraint satisfied in $\mathcal{X}_f$),*

A3: *$\mathcal{X}_f$ is positively invariant for the system, $x^+ = f(x, \kappa_f(x))$,*

A4: *$[\overset{*}{F} + \ell](x, \kappa_f(x)) \leq 0$, $\forall x \in \mathcal{X}_f$ ($F(\cdot)$ is a local Lyapunov function).*

Then

$$[\overset{*}{V_N^0} + \ell](x, \kappa_N(x)) \leq 0$$

for all $x \in \mathcal{X}_N$, the (compact, convex) set of states steerable to $\mathcal{X}_f$ by an admissible control in time N or less. Also $\mathcal{X}_N$ is positively invariant for the closed-loop system $x^+ = f(x, \kappa_N(x))$ where $\kappa_N(\cdot)$ is the model predictive control law. ●

Corollary 1 *Suppose $Q > 0$ and $R > 0$, that $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ satisfy **A1–A4** and that, in addition, there exists a finite c such that $F(x) \leq c|x|^2$ for all $x \in \mathcal{X}_f$. Then the origin is exponentially stable for the closed-loop system $x^+ = f(x, \kappa_N(x))$ with a region of attraction $\mathcal{X}_N$.*

Proof: Since $Q > 0$ it follows from Theorem 1 that there exists a finite positive constant a such that

$$V_N^0(x) \geq a|x|^2, \ \forall x \in \mathcal{X}_N$$

and

$$\overset{*}{V_N^0}(x, \kappa_N(x)) \leq -a|x|^2, \ \forall x \in \mathcal{X}_N$$

Next $V_N^0(x) \leq F(x)$ for all $x \in \mathcal{X}_f$ [4]. This is easily shown. Let x be an arbitrary point in $\mathcal{X}_f$ and let $\{x^f(k;x), k = 0, 1, 2, \ldots\}$ be the state sequence resulting from initial state x and controller $\kappa_f(\cdot)$. Then, by **A4**

$$F(x) \geq \sum_{k=0}^{N-1} \ell(x^f(k;x), \kappa_f(x^f(k;x))) + F(x^f(N;x))$$

where (by **A3**) $x^f(k;x) \in \mathcal{X}_f$ for all $k = 0, 1, \ldots, N$ and (by **A2**) $\kappa_f(x^f(k;x)) \in \Omega$ for all $k = 0, 1, \ldots, N-1$. But, by optimality (since $x^f(N;x) \in \mathcal{X}_f$),

$$V_N^0(x) \leq \sum_{k=0}^{N-1} \ell(x^f(k;x), \kappa_f(x^f(k;x))) + F(x^f(N;x))$$

Hence $V_N^0(x) \leq F(x) \leq c|x|^2$ for all $x \in \mathcal{X}_f$. Exponential stability, with a region of attraction $\mathcal{X}_N$, follows. ■

3.2 Regional characterisation of the value function

We review here some recent results [2] that show that the nonlinear controller $\kappa_{n\ell}(\cdot)$ defined by

$$\kappa_{n\ell}(x) := \mathrm{sat}(-Kx) = -\mathrm{sat}(Kx) \tag{3.12}$$

is optimal in a non-trivial region of the state-space (in the sense that the region includes the output admissible set, which is defined in the sequel). In (3.12), the vector K is the optimal gain for the *unconstrained* infinite-horizon problem, which is computed from the (unique positive semi-definite) solution of the algebraic Riccati equation

$$P = A^T PA + Q - K^T \bar{R} K, \tag{3.13}$$

where

$$K := \bar{R}^{-1} B^T PA, \quad \bar{R} := R + B^T PB. \tag{3.14}$$

In the sequel we consider both a linear controller $u = -Kx$ and a nonlinear controller $u = -\mathrm{sat}(Kx)$. The closed-loop satisfies $x^+ = \phi_\ell(x)$ when the linear controller is used and $x^+ = \phi_{n\ell}(x)$ when the nonlinear controller is employed, where the mappings $\phi_\ell : \mathbb{R}^n \to \mathbb{R}^n$ and, $\phi_{n\ell} : \mathbb{R}^n \to \mathbb{R}^n$ are defined by

$$\phi_\ell(x) := A_K x, \quad A_K := A - BK, \tag{3.15}$$

$$\phi_{n\ell}(x) := Ax - B\ \mathrm{sat}(Kx) \tag{3.16}$$

For all $i \in \{1, 2, \ldots, N\}$ ($N \geq 1$ an integer) the function $\delta_i(\cdot)$ is defined by

$$\delta_i(x) := Kx - \mathrm{sat}_{\Delta_i}(Kx) \tag{3.17}$$

where the saturation bounds Δ_i are defined by

$$\Delta_1 \quad := \quad 1 \tag{3.18}$$

$$\Delta_i \quad := \quad 1 + \sum_{j=0}^{i-2} |KA^j B|, \quad i = 2, 3, \ldots, N \tag{3.19}$$

The set $X_0 := \mathbb{R}^n$, and for each $i \in \{1, 2, \ldots, N-1\}$, the set $X_i \subset \mathbb{R}^n$ is defined by

$$X_i := \{x \mid \delta_i(A^{i-1} A_K x) = 0\} = \{x \mid |\bar{K}_i x| \leq \Delta_i\} \tag{3.20}$$

where

$$\bar{K}_i := KA^{i-1} A_K. \tag{3.21}$$

We also require the sets Y_i ($i \in \{1, 2, \ldots, N\}$) and Z_N defined by

$$Y_i \quad := \quad X_0 \cap X_1 \ldots \cap X_{i-1}, \tag{3.22}$$

$$Z_1 \quad := \quad Y_1 = \mathbb{R}^n, \tag{3.23}$$

$$Z_N \quad := \quad \{x \mid \phi_{n\ell}^k(x) \in Y_{N-k}, k = 0, 1, \ldots, N-2\}, \text{ for } N \geq 2. \tag{3.24}$$

We can now state the main results of [2]:

Lemma 1 *For any $i \in \{1, 2, \ldots, N-1\}$ define the functions $\phi_{n\ell}(\cdot)$ and $\delta_i(\cdot)$, $\delta_{i+1}(\cdot)$ as in (3.16) and (3.17) respectively, and the set X_i as in (3.20). Then:*

$$\delta_i(A^{i-1}\phi_{n\ell}(x))^2 = \delta_{i+1}(A^i x)^2, \qquad \text{for all} \quad x \in X_i. \tag{3.25}$$

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Theorem 2 *The optimal value function $V_N^0(\cdot)$ for problem $\mathcal{P}_N$ with $F(x) := x^T P x$ and $\mathcal{X}_f := I\!R^n$ satisfies*

$$V_N^0(x) = J^N(x) := x^T P x + \bar{R} \sum_{k=1}^{N} \delta_k (A^{k-1}x)^2, \ \forall x \in Z_N \tag{3.26}$$

and the optimal control law $\kappa_N(\cdot)$ satisfies

$$\kappa_N(x) = \kappa_{n\ell}(x) = -\mathrm{sat}(Kx), \ \forall x \in Z_N \tag{3.27}$$

where the set Z_N is defined by (3.24). ●

Since the functions $x \mapsto \delta_k(A^{k-1}x)^2$ are convex, so is the value function $V_N^0(\cdot)$.

4 Terminal conditions

4.1 Standard specification of $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$

A triple $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ satisfying **A1**–**A4** and $F(x) \leq c|x|^2$ for all $x \in \mathcal{X}_f$ ensures exponential stability as shown in §3.1 above. A useful choice of terminal conditions [9, 10] for the problem considered is to choose $F(\cdot)$ to be the value function $V_{uc}^0(\cdot)$ for the *unconstrained* infinite-horizon optimal control problem $\mathcal{P}_{uc}(x)$ for the same system (2.1), defined as

$$\mathcal{P}_{uc}(x): \quad V_{uc}^0(x) = \min_{\mathbf{u}} V_{uc}(x, \mathbf{u}) \tag{4.1}$$

with cost

$$V_{uc}(x, \mathbf{u}) := \sum_{k=0}^{\infty} \ell(x(k), u(k)) \tag{4.2}$$

where $\ell(x, u) = |x|_Q^2 + |u|_R^2$ as before. (Note that $\mathcal{P}_{uc}(\cdot)$ does not have either a terminal cost nor a terminal constraint; both are irrelevant since, if a solution to the problem exists, $x^0(k; x) \to 0$ as $k \to \infty$.)

Thus, in the *constrained* optimisation problem $\mathcal{P}_N(\cdot)$ (3.1), solved at each time instant in model predictive control, the terminal cost function used in this case is

$$F(x) := V_{uc}^0(x) = x^T P x \tag{4.3}$$

where $P > 0$ is the (unique positive semi-definite) solution of the algebraic Riccati equation (3.13)–(3.14).

The local controller is defined by

$$\kappa_f(x) := -Kx \tag{4.4}$$

where K is computed from (3.14), and is, therefore, the optimal controller for the unconstrained infinite-horizon problem $\mathcal{P}_{uc}(\cdot)$. The set $\mathcal{X}_f$ is usually taken to be the maximal output admissible set $\mathcal{O}_\infty$ defined in [3], i.e.

$$\mathcal{O}_\infty := \{x \mid KA_K^j x \in \Omega, \ j = 0, 1, \ldots\}. \tag{4.5}$$

An interesting consequence of this choice for $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ is that $V_\infty^0(x) = F(x)$ for all x in $\mathcal{X}_f$ and that $V_N^0(x) = V_\infty^0(x)$ for all $x \in \mathcal{X}_N$ such that the terminal constraint is not active (i.e. $x^0(N; x)$ lies in the interior of $\mathcal{X}_f$); if N is so chosen, the terminal constraint may be omitted from $\mathcal{P}_N(\cdot)$.

4.2 New specification of $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$

It is the purpose of this paper to propose a *larger* terminal constraint set $\mathcal{X}_f$, thus simplifying optimisation (or reducing N in those variants that omit the terminal constraint from the optimal control problem but increase the horizon N until this constraint is satisfied). To this end we employ the results described in §3.2; namely, the regional characterisation of the value function $J^N(x)$ (3.26) when the optimal control law $\kappa_{n\ell}(x)$ (3.27) is employed for states x in a region Z_N of the state space. We show in this section that the new triple $(F(\cdot), \mathcal{X}_f, \kappa_f)$, obtained using these elements, satisfies conditions **A1**–**A4** of Theorem 1 and constitutes an improvement over previous results (cf. §4.1).

It can be readily seen from (3.27) that the control law $\kappa_{n\ell}(\cdot)$ satisfies **A2**. Our problem then reduces to find a set $\mathcal{X}_f$ that satisfies **A1**, is positively invariant under the control law $\kappa_{n\ell}(\cdot)$ and in which $J^N(\cdot)$ is a local Lyapunov function.

Definition 1 *Define the sets $\bar{X}_N$, $\bar{Y}_N$, $\bar{Z}_N \subset \mathbb{R}^n$, for $N \geq 1$, as*

$$\bar{X}_N := \{x \mid \delta_N(A^{N-1}\phi_{n\ell}(x)) = 0\} = \{x \mid |KA^{N-1}\phi_{n\ell}(x)| \leq \Delta_N\}, \tag{4.6}$$

$$\bar{Y}_N := Y_N \cap \bar{X}_N \cap \mathcal{D}_S = X_0 \cap \ldots X_{N-1} \cap \bar{X}_N \cap \mathcal{D}_S, \tag{4.7}$$

and

$$\bar{Z}_N := \{x \mid \phi^k_{n\ell}(x) \in \bar{Y}_N, k = 0, 1, 2, \ldots\}, \tag{4.8}$$

where the set Y_N is as in (3.22) and $\mathcal{D}_S$ is a 'design set', used to ensure compactness of $\bar{Y}_N$. In the case when $Y_N \cap \bar{X}_N$ is compact, $\mathcal{D}_S$ can be chosen to be $\mathcal{D}_S = \mathbb{R}^n$; otherwise, $\mathcal{D}_S$ is chosen to be an arbitrarily large compact set such that

$$\mathcal{O}_\infty \subset \mathcal{D}_S. \tag{4.9}$$

●

The set $\bar{Z}_N$ is a candidate for $\mathcal{X}_f$, but for this use it is necessary that it be *finitely determined.* To establish this we require:

Proposition 1 *(i) The set $\bar{Z}_N$ is positively invariant for the system $x^+ = \phi_{n\ell}(x)$. (ii) $\bar{Z}_N \subset \bar{Y}_N$. (iii) $\bar{Z}_N$ is compact and contains the origin in its interior.*

Proof: (i) This follows from the definition (4.8) of $\bar{Z}_N$ if $\bar{Z}_N$ is not empty. Let $c > 0$ be such that the level set $\mathcal{L} := \{x \mid x^T P x \leq c\} \subset \mathcal{O}_\infty \cap \bar{Y}_N$; since both $\mathcal{O}_\infty$ [3] and $\bar{Y}_N$ contain the origin in their interiors and since $P > 0$, such a c exists. Since $\kappa_{n\ell}(x) = -Kx$ and $\phi_{n\ell}(x) = A_K x$ in $\mathcal{O}_\infty$, and since $\mathcal{L}$ is positively invariant for $x^+ = A_K x = \phi_{n\ell}(x)$, it follows that $\mathcal{L} \subset \bar{Z}_N$. But $\mathcal{L}$ contains the origin in its interior (since $P > 0$). (ii) From definition (4.8), $x \in \bar{Z}_N$ implies $x \in \bar{Y}_N$. (iii) Since $\bar{Y}_N$ is compact and $\phi_{n\ell}(\cdot)$ is continuous, each set $W_k := \{x \mid \phi^k_{n\ell}(x) \in \bar{Y}_N\}$, $k = 0, 1, \ldots$, is compact. Hence $\bar{Z}_N$ is compact. ■

Proposition 2 $\bar{Z}_N \subset Z_N$.

Proof: Notice first, from (3.22) and (4.7), that $\bar{Y}_N \subset Y_i$, $i = 1, 2, \ldots, N$. It follows from definition (4.8) that $x \in \bar{Z}_N$ implies $\phi^k_{n\ell}(x) \in \bar{Y}_N$, for all k. Hence, $\phi^k_{n\ell}(x) \in Y_i$, for all k and $i = 2, 3, \ldots, N$, and we conclude from (3.24) that $x \in Z_N$. ■

Proposition 3 *For all $x \in \bar{Z}_N$:*

$$[\overset{*}{J^N} + \ell](x, \kappa_{n\ell}(x)) = 0. \tag{4.10}$$

Proof: Making use of (3.11), (3.13), (3.14), (3.25), (3.26), (4.6), (4.7), and the fact that $\bar{Z}_N \subset \bar{Y}_N$ (Proposition 1) and $\bar{Z}_N \subset Z_N$ (Proposition 2), we obtain, for $x \in \bar{Z}_N$,

$$\begin{aligned}
[\overset{*}{J^N} + \ell](x, \kappa_{n\ell}(x)) &= J^N(\phi_{n\ell}(x)) + \ell(x, \kappa_{n\ell}(x)) - J^N(x) \\
&= J^N(x) + \bar{R}\,\delta_N (A^{N-1}\phi_{n\ell}(x))^2 - J^N(x) \\
&= \bar{R}\,\delta_N (A^{N-1}\phi_{n\ell}(x))^2 = 0.
\end{aligned}$$

■

For all $j = 1, 2, \ldots$, let W_j be defined by

$$W_j := \{x \mid \phi^i_{n\ell}(x) \in \bar{Y}_N \text{ for } i = 0, 1, \ldots, j-1 \text{ and } \phi^j_{n\ell}(x) \in \mathcal{L}\} \tag{4.11}$$

where $\mathcal{L}$ is defined in the proof of Proposition 1.

Proposition 4 *There exists an integer $i^\star$ such that $\bar{Z}_N = W_{i^\star}$; i.e., $\bar{Z}_N$ is finitely determined.*

Proof: (i) Let

$$\max\{J^N(x) \mid x \in \bar{Y}_N\} = c_1 < \infty$$

¿From (4.10) we have that

$$\overset{*}{J^N}(x, \kappa_{n\ell}(x)) = -\ell(x, \kappa_{n\ell}(x)) \le -|x|^2_Q \le -c_2|x|^2, \quad \forall\, x \in \bar{Z}_N. \tag{4.12}$$

There exists a $c_3 \in (0, \infty)$ such that

$$\overset{*}{J^N}(x, \kappa_{n\ell}(x)) \le -c_3,$$

for all $x \in \bar{Z}_N \cap \text{closure}(\mathcal{L}^c)$. Hence for all $x \in \bar{Z}_N$ there exists an integer $i^\star \le c_1/c_3$ such that $\phi^{i^\star}_{n\ell}(x) \in \mathcal{L}$. Hence $x \in \bar{Z}_N$ implies $x \in W_{i^\star}$.
(ii) Suppose $x \in W_{i^\star}$ so that $\phi^i_{n\ell}(x) \in \bar{Y}_N$ for $i = 0, 1, \ldots, i^\star - 1$ and $\phi^{i^\star}_{n\ell} \in \mathcal{L}$. Since $\mathcal{L}$ is positively invariant for $x^+ = \phi_{n\ell}(x)$, $\phi^j_{n\ell}(x) \in \mathcal{L} \subset \bar{Y}_N$ for $j = i^\star, i^\star+1, i^\star+2, \ldots$. Hence $x \in \bar{Z}_N$. ■

¿From the definition (4.8) it is clear that $\bar{Z}_N$ is the *maximal positively invariant set* in $\bar{Y}_N$ for the closed-loop system $x^+ = \phi_{n\ell}(x)$ and, hence, $\mathcal{X}_f := \bar{Z}_N$ satisfies **A3**.

We now establish that $\mathcal{O}_\infty$ is a subset of $\bar{Z}_N$.

Proposition 5 $\mathcal{O}_\infty \subset \bar{Z}_N$.

Proof: By definition $\bar{Z}_N$ is the maximal positively invariant set in $\bar{Y}_N$ for the closed-loop system $x^+ = \phi_{n\ell}(x)$. The set $\mathcal{O}_\infty$ is also a positively invariant set for $x^+ = \phi_{n\ell}(x)$ (since $\phi_{n\ell}(\cdot) = \phi_\ell(\cdot)$ in $\mathcal{O}_\infty$). It suffices, therefore, to establish that $\mathcal{O}_\infty \subset \bar{Y}_N$; i.e., that $\mathcal{O}_\infty \subset X_i$ for $i = 1, 2, \ldots, N-1$ and $\mathcal{O}_\infty \subset \bar{X}_N$, since $\mathcal{O}_\infty \subset \mathcal{D}_S$ by design (cf. (4.9)). Assume, therefore, that $x \in \mathcal{O}_\infty$, so that

$$|KA^j_K x| \le 1, \quad j = 0, 1, \ldots \tag{4.13}$$

For any $i \in \{1, 2, \ldots, N\}$

$$\begin{aligned}
A_K^i &= (A - BK)A_K^{i-1} = AA_K^{i-1} - BKA_K^{i-1} \\
&= A(A - BK)A_K^{i-2} - BKA_K^{i-1} = A^2A_K^{i-2} - ABKA_K^{i-2} - BKA_K^{i-1} \\
&= A^2(A - BK)A_K^{i-3} - ABKA_K^{i-2} - BKA_K^{i-1} \\
&= A^3A_K^{i-3} - A^2BKA_K^{i-3} - ABKA_K^{i-2} - BKA_K^{i-1} \\
&\vdots \\
&= A^{i-1}A_K - \sum_{j=0}^{i-2} A^jBKA_K^{i-1-j}
\end{aligned}$$

which implies

$$KA^{i-1}A_Kx = KA_K^ix + \sum_{j=0}^{i-2} KA^jBKA_K^{i-1-j}x \tag{4.14}$$

¿From (4.13) and (4.14), we obtain the inequality

$$\begin{aligned}
|KA^{i-1}A_Kx| &\leq |KA_K^ix| + \sum_{j=0}^{i-2} |KA^jB||KA_K^{i-1-j}x| && (4.15) \\
&\leq 1 + \sum_{j=0}^{i-2} |KA^jB| = \Delta_i && (4.16)
\end{aligned}$$

This implies $x \in X_i$ for $i = 1, 2, \ldots, N-1$ (cf. (3.20)). To show that $\mathcal{O}_\infty \subset \bar{X}_N$ notice, from (4.6), that $\bar{X}_N$ can be written as the union of three sets:

$$\bar{X}_N = V_1 \cup V_2 \cup V_3, \tag{4.17}$$

where

$$\begin{aligned}
V_1 &:= \{x \mid |Kx| \leq 1\} \cap \{x \mid |KA^{N-1}A_Kx| \leq \Delta_N\}, \\
V_2 &:= \{x \mid Kx < -1\} \cap \{x \mid |KA^{N-1}(Ax + B)| \leq \Delta_N\}, \\
V_3 &:= \{x \mid Kx > 1\} \cap \{x \mid |KA^{N-1}(Ax - B)| \leq \Delta_N\}.
\end{aligned}$$

Then, it follows from (4.13) and (4.16) that $x \in \mathcal{O}_\infty$ implies $x \in V_1$ and, hence, $x \in \bar{X}_N$. ■

Example 1 *In this example, the relative size of the sets $\mathcal{O}_\infty$, $\bar{Y}_N$ and $\bar{Z}_N$ is illustrated. Consider the system $x^+ = Ax + Bu$, $y = Cx$ with*

$$A = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.4 \\ 0.08 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix},$$

which is the zero-order hold discretisation, with a sampling period of 0.4 sec., of the double integrator $\dot{x}_1 = u$, $\dot{x}_2 = x_1$, $y = x_2$. In Eqs. (3.5)–(3.6) we take: $Q = I_{2\times 2}$ and $R = 0.25$. The matrix P and the gain K were computed from (3.13)–(3.14).

In Figure 1 we show the set $\bar{Y}_N$, the maximal output admissible set $\mathcal{O}_\infty$ and an 'estimate' of the set $\bar{Z}_N$. In this example Y_N is compact and $Y_N \subset \bar{X}_N$. Therefore, we take $\mathcal{D}_S = \mathbb{R}^n$ in (4.7) and obtain

$$\bar{Y}_N = Y_N = \cap_{i=0}^{N-1} X_i.$$

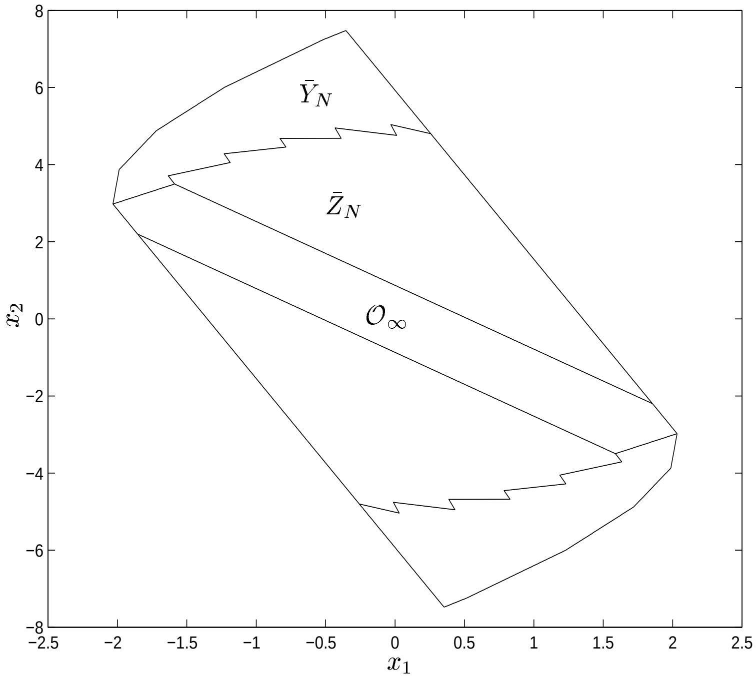

Figure 1: Set boundaries for the example. (In the case of $\bar{Z}_N$ an *estimate* obtained *numerically* is shown. Note that $\mathcal{O}_\infty \subset \bar{Z}_N \subset \bar{Y}_N$.)

The estimate of $\bar{Z}_N$ was obtained numerically by gridding the set $\bar{Y}_N$ and checking the condition in (4.8) for each point of the grid. In the implementation of the MPC algorithm, the test $x^0(N;x) \in \mathcal{X}_f$ is performed 'on-line' (see §4.3 below) and, thus, an 'explicit' characterisation of $\mathcal{X}_f := \bar{Z}_N$ is not required. We have included an 'estimate' of $\bar{Z}_N$ in the figure for illustration purposes. As proven in Proposition 4, the set $\bar{Z}_N$ is finitely determined and, in this example, the test $\phi^k_{n\ell}(x) \in \bar{Y}_N$ (cf. (4.8)) is stopped for k such that $\phi^k_{n\ell}(x) \in \mathcal{O}_\infty$. As the figure shows, the new terminal constraint set $\bar{Z}_N$ is considerably larger than $\mathcal{O}_\infty$. ●

We now state the main result of this paper.

Theorem 3 *The triple $(F(\cdot), \mathcal{X}_f, \kappa_f(\cdot))$ where $F(\cdot) := J^N(\cdot)$, $\mathcal{X}_f := \bar{Z}_N$ and $\kappa_f(\cdot) := \kappa_{n\ell}(\cdot)$, satisfies conditions **A1–A4**.*

Proof: (i) It is established in Proposition 1 above that $\mathcal{X}_f := \bar{Z}_N$ is closed and contains the origin in its interior, thus satisfying **A1**.
(ii) $\kappa_f(\cdot) := \kappa_{n\ell}(\cdot)$ satisfies **A2** by definition (cf. (3.27)).
(iii) That $\mathcal{X}_f := \bar{Z}_N$ is positively invariant for the system $x^+ = f(x, \kappa_f(x)) = \phi_{n\ell}(x)$ is established in Proposition 1 above.
(iv) We have shown in Proposition 3 (cf. (4.10)) that $F(\cdot) := J^N(\cdot)$ satisfies

$$[\overset{*}{F} + \ell](x, \kappa_{n\ell}(x)) = 0$$

for all $x \in \mathcal{X}_f$, thus satisfying **A4**. ■

4.3 Implementation of the MPC algorithm

Since $\bar{Z}_N$ is not defined by linear inequalities and $J^N(\cdot)$ is not quadratic, problem $\mathcal{P}_N(\cdot)$ with $F(\cdot) = J^N(\cdot)$, $\mathcal{X}_f = \bar{Z}_N$ and $\kappa_f(\cdot) = \kappa_{n\ell}(\cdot)$ is not a quadratic program. Indeed it is not even necessarily a convex program since there is no guarantee that $\bar{Z}_N$ is convex. (In general $\bar{Z}_N$ is nonconvex, as Figure 1 illustrates.) However, because $\bar{Z}_N$ is finitely specified, the variant of model predictive control in which the terminal constraint is omitted from $\mathcal{P}_N(\cdot)$ and N is chosen (either *a priori* or online) to ensure that the terminal constraint is satisfied (despite its omission from $\mathcal{P}_N(\cdot)$) is easily implemented. The resultant problem is convex because $F(\cdot)$ is convex and positive definite and may be solved using convex programming or, indeed, conventional non-linear programming (e.g., [8]).

5 Conclusions

We have shown how to obtain new terminal ingredients $F(\cdot)$, $\mathcal{X}_f$, $\kappa_f(\cdot)$ (for the optimal control problem employed in model predictive control) that ensure closed-loop stability. The ingredients provide a larger terminal constraint set than that provided by previous approaches thus facilitating online solution of the optimal control problem. Examples show that the new constraint set $\mathcal{X}_f$ is larger than the output admissible set $\mathcal{O}_\infty$ conventionally employed.

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